

FRACTIONAL INTEGRAL OPERATORS FOR L^1 AND WEIGHTED L^1 VECTOR FIELDS AND ITS APPLICATIONS

ZHIBING ZHANG

ABSTRACT. This paper studies fractional integral operators for vector fields in some subspaces of L^1 and weighted L^1 . Using the estimates on fractional integral operators and Stein-Weiss inequalities, we can give a new proof for a class of Caffarelli-Kohn-Nirenberg inequalities and establish new div-curl inequalities for vector fields.

1. INTRODUCTION

For $x \in \mathbb{R}^n$ and $r > 0$. Let $B_r(x)$ denote the open ball centered at x of radius r . Let f be a locally integrable function. For any $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal function $Mf(x)$ of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy.$$

For $0 < \lambda < n$, the fractional integral operator I_λ is defined by

$$I_\lambda f(x) = (-\Delta)^{-\frac{\lambda}{2}} f(x) = C_{n,-\lambda} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy,$$

which is also called the Riesz potential. We recall classical results on the Riesz potential, see [18, Lemma 3.1.1, Theorem 3.1.1].

Theorem 1.1 (Hardy-Littlewood-Sobolev). *Let $0 < \lambda < n$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{n}$.*

(1) *For $x \in \mathbb{R}^n$, we have*

$$|I_\lambda f(x)| \leq C \|f\|_{L^p}^{\frac{\lambda p}{n}} (Mf(x))^{1-\frac{\lambda p}{n}}.$$

(2) *If $p > 1$, then $\|I_\lambda f\|_{L^q} \leq C \|f\|_{L^p}$.*

(3) *If $p = 1$, then for all $t > 0$, there exists a constant $C = C(n, q, \lambda)$ such that*

$$t(|\{x \in \mathbb{R}^n : |I_\lambda f(x)| > t\}|)^{\frac{1}{q}} \leq C \|f\|_{L^1}.$$

Consequently, by Kolmogorov's theorem (see [18, Theorem 1.3.3]), for any set $E \subseteq \mathbb{R}^n$ with finite measure, we have $I_\lambda f \in L^r(E)$ and

$$\|I_\lambda f\|_{L^r(E)} \leq C \left(\frac{q}{q-r} \right)^{\frac{1}{r}} |E|^{\frac{1}{r}-\frac{1}{q}} \|f\|_{L^1}, \text{ where } 0 < r < q.$$

2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 35F35.

Key words and phrases. Hardy-Littlewood-Sobolev inequality, Stein-Weiss inequality, fractional integral operators, L^1 vector fields, div-curl inequalities.

It is well-known that I_λ is not a bounded operator from L^1 to L^q . But if we work in some subsets of L^1 , such as $(L^1 \cap L \log^+ L)(\mathbb{R}^n)$ and Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, we can expect substitute results.

Stein and Weiss [24] established a doubly weighted generalization of Hardy-Littlewood-Sobolev inequality as follows.

Theorem 1.2 (Stein-Weiss). *Let $0 < \lambda < n$, $1 < p < \infty$, $\alpha < \frac{n}{p'}$, $\beta < \frac{n}{q}$, $\alpha + \beta \geq 0$, and $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta - \lambda}{n}$. If $p \leq q < \infty$, then there exists a constant C , independent of f , such that*

$$\| |x|^{-\beta} I_\lambda f \|_{L^q} \leq C \| |x|^\alpha f \|_{L^p}.$$

De Nápoli, Drelichman and Durán [11] found that if we reduce ourselves to radially symmetric functions, Theorem 1.2 holds for a wider range of exponents. For radially symmetric functions, they pointed out that if $p > 1$, Theorem 1.2 holds for $\alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$; if $p = 1$, Theorem 1.2 holds for $\alpha + \beta > (n-1)(\frac{1}{q} - 1)$.

Considering that we are interested in the extreme case $p = 1$ of Stein-Weiss inequality, we turn to end-point estimates for L^1 vector fields, which was pioneered by Bourgain and Brezis [3]. For any vector-valued function $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, we denote by $\mathbf{u} = (-\Delta)^{-1} \mathbf{f} = \Gamma * \mathbf{f}$ the Newtonian potential of \mathbf{f} . For $n \geq 2$, Bourgain and Brezis [4, 5] proved that

$$\| \nabla \mathbf{u} \|_{L^{n'}(\mathbb{R}^n)} \leq C (\| \mathbf{f} \|_{L^1(\mathbb{R}^n)} + \| \operatorname{div} \mathbf{f} \|_{\dot{W}^{-2, n'}(\mathbb{R}^n)}), \quad (1.1)$$

which is equivalent to

$$\| \mathbf{f} \|_{\dot{W}^{-1, n'}(\mathbb{R}^n)} \leq C (\| \mathbf{f} \|_{L^1(\mathbb{R}^n)} + \| \operatorname{div} \mathbf{f} \|_{\dot{W}^{-2, n'}(\mathbb{R}^n)}). \quad (1.2)$$

Hence, for $n \geq 3$, it holds that

$$\| \mathbf{u} \|_{L^{\frac{n}{n-2}}(\mathbb{R}^n)} \leq C (\| \mathbf{f} \|_{L^1(\mathbb{R}^n)} + \| \operatorname{div} \mathbf{f} \|_{\dot{W}^{-2, n'}(\mathbb{R}^n)}).$$

For $n = 2$, they also proved

$$\| \mathbf{u} \|_{L^\infty(\mathbb{R}^2)} \leq C \| \mathbf{f} \|_{L^1(\mathbb{R}^2)} \text{ for } \mathbf{f} \in L^1(\mathbb{R}^2) \text{ with } \operatorname{div} \mathbf{f} = 0.$$

Moreover, for divergence-free vector fields, Chanillo and Yung [9] pointed out the following result, which is also a consequence of [28, Proposition 2.1] and embedding theorem of homogeneous Triebel-Lizorkin spaces.

Theorem 1.3 (Chanillo-Yung). *Let $n \geq 2$, $0 < \lambda < n$, $\frac{1}{q} = 1 - \frac{\lambda}{n}$, $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{div} \mathbf{f} = 0$. Then*

$$\| \mathbf{f} \|_{\dot{W}^{-\lambda, q}(\mathbb{R}^n)} \leq C \| \mathbf{f} \|_{L^1(\mathbb{R}^n)},$$

which is equivalent to

$$\| I_\lambda \mathbf{f} \|_{L^q(\mathbb{R}^n)} \leq C \| \mathbf{f} \|_{L^1(\mathbb{R}^n)}.$$

Maz'ya [20] established weighted inequalities related to (1.1) as follows:

- (1) Let $1 \leq q < n'$, $\beta = 1 - n(1 - \frac{1}{q})$ and $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} \mathbf{f} = 0$. Then it holds that

$$\left\| \frac{\nabla \mathbf{u}}{|x|^\beta} \right\|_{L^q(\mathbb{R}^n)} \leq C \| \mathbf{f} \|_{L^1(\mathbb{R}^n)}. \quad (1.3)$$

- (2) Let $1 < q < n'$, $\beta = 1 - n(1 - \frac{1}{q})$, $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Then it holds that

$$\left\| \frac{\nabla \mathbf{u}}{|x|^\beta} \right\|_{L^q(\mathbb{R}^n)} \leq C(\|\mathbf{f}\|_{L^1(\mathbb{R}^n)} + \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1(\mathbb{R}^n)}). \quad (1.4)$$

Soon after, Bousquet and Mironescu [6] gave a short proof of Maz'ya's result and got a little improvement. They found that the inequality (1.4) holds also for $q = 1$. In fact, using Leray decomposition and a similar trick used in the proof of Theorem 1.4, we can prove that the inequality (1.3) and the result of Bousquet and Mironescu are equivalent, see Remark 3.9. Inspired by these inequalities, we try to extend the range of exponents that Theorem 1.2 holds for in the case of weighted vector fields. We introduce a family of fractional integral operators T_λ defined by

$$T_\lambda f(x) = K * f(x) = \int_{\mathbb{R}^n} K(x-y) f(y) dy,$$

where the kernel $K(x)$ satisfies

$$(i) |K(x)| \leq C|x|^{\lambda-n}, \text{ if } |x| \neq 0; (ii) |K(x-y) - K(x)| \leq C \frac{|y|}{|x|^{n+1-\lambda}}, \text{ if } |y| \leq \frac{|x|}{2}. \quad (1.5)$$

We give two examples of such kernels: $K(x) = |x|^{\lambda-n}$ and $K(x) = \frac{x_j}{|x|^{n-\lambda+1}}$. If you want to check and learn more, see Proposition 3.1. Hence, $(-\Delta)^{-1} f$ is of the type $T_\lambda f$ for $n > 2$ and $\nabla(-\Delta)^{-1} f$ is of the type $T_\lambda f$ for $n \geq 2$.

In this paper, we establish end-point estimates for the fractional integral operators for vector fields in some subspaces of L^1 via the fractional Laplacian, and we adopt the skills developed by Maz'ya, Bousquet, Mironescu to deal with the fractional integral operators for vector fields in some subspaces of weighted L^1 . As a result, we find some spaces of vector-valued functions make Stein-Weiss inequalities hold for the case $p = 1$. We now state our main results.

Theorem 1.4. *Let $n \geq 2$, $0 < \lambda < n$, $\frac{1}{q} = 1 - \frac{\lambda}{n}$, $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Suppose that $K(x)$ satisfies the conditions in (1.5). Then*

$$\|T_\lambda \mathbf{f}\|_{L^q} \leq C(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}). \quad (1.6)$$

Theorem 1.5. *Let $n \geq 2$, $0 < \lambda < n$, $\alpha < 1$, $\beta < \frac{n}{q}$, $\alpha + \beta > 0$, $\frac{1}{q} = 1 + \frac{\alpha+\beta-\lambda}{n}$. Suppose that $K(x)$ satisfies the conditions in (1.5). If $1 \leq q < \infty$, then*

$$\||x|^{-\beta} T_\lambda \mathbf{f}\|_{L^q} \leq C(\||x|^\alpha \mathbf{f}\|_{L^1} + \||x|^\alpha \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}). \quad (1.7)$$

This paper is organized as follows. In Section 2, we give some notations that have appeared in the context. Section 3 shows the proofs of Theorem 1.4 and Theorem 1.5. In Section 4, applying the new inequalities and classical Stein-Weiss inequalities, we give a new proof to Hardy inequality as well as a class of Caffarelli-Kohn-Nirenberg inequalities and obtain new div-curl inequalities for vector fields.

2. NOTATIONS AND DEFINITIONS

Let $\Omega \subseteq \mathbb{R}^n$. The notation $\mathcal{D}(\Omega, \mathbb{R}^n)$ denotes the space of n -dimensional vector-valued functions that are infinitely differentiable and have compact supports in Ω . Let $\mathcal{D}'(\Omega, \mathbb{R}^n)$ denote the dual space of $\mathcal{D}(\Omega, \mathbb{R}^n)$. We denote Zygmund space by $L \log^+ L(E)$, which is defined as follows: if the function f satisfies

$$\int_E |f(x)| \log^+ |f(x)| dx < \infty,$$

then we say $f \in L \log^+ L(E)$, where $E \subseteq \mathbb{R}^n$ and $\log^+ t$ is defined by

$$\log^+ t = \begin{cases} \log t, & t > 1, \\ 0, & 0 \leq t \leq 1. \end{cases}$$

The Riesz transform R_j is defined by

$$R_j f(x) = C_{n,p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy, \quad 1 \leq j \leq n.$$

It is not difficult to see that $(R_1 f, R_2 f, \dots, R_n f) = \nabla(-\Delta)^{-\frac{1}{2}} f$. The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$\mathcal{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : R_j f \in L^1(\mathbb{R}^n), 1 \leq j \leq n\}$$

and its norm is given by

$$\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \sum_{j=1}^n \|R_j f\|_{L^1}.$$

Let \mathcal{S} stand for the Schwartz space of rapidly decreasing C^∞ functions in \mathbb{R}^n . When $0 < \alpha < 2$ and $f \in \mathcal{S}$, $(-\Delta)^{\frac{\alpha}{2}} f$ is defined by a singular integral

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_{n,\alpha,p.v.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy.$$

By [12, Lemma 3.2], the singularity of the above integral can be removed by a weighted second order differential quotient, i.e.,

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = -\frac{1}{2} C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+\alpha}} dy.$$

There are several equivalent definitions of the fractional Laplacian operator, see [12] or [22]. The space $\mathbf{H}^p(\text{div}, \Omega)$ is defined by

$$\mathbf{H}^p(\text{div}, \Omega) = \{\mathbf{v} \in L^p(\Omega, \mathbb{R}^n) : \text{div } \mathbf{v} \in L^p(\Omega)\}$$

and is provided with the norm

$$\|\mathbf{v}\|_{\mathbf{H}^p(\text{div}, \Omega)} = \|\mathbf{v}\|_{L^p(\Omega)} + \|\text{div } \mathbf{v}\|_{L^p(\Omega)}.$$

In particular, the space $\mathbf{H}^p(\text{div } 0, \Omega)$ is given by

$$\mathbf{H}^p(\text{div } 0, \Omega) = \{\mathbf{v} \in L^p(\Omega, \mathbb{R}^n) : \text{div } \mathbf{v} = 0\}.$$

We denote by $\mathbf{H}_0^p(\operatorname{div}, \Omega)$ the closure of $C_c^\infty(\Omega, \mathbb{R}^n)$ in $\mathbf{H}^p(\operatorname{div}, \Omega)$. It is well-known that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$. Proposition 3.10 implies that $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$ has the same property, i.e., $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n) = \mathbf{H}_0^p(\operatorname{div}, \mathbb{R}^n)$. We denote the fundamental solution of $-\Delta$ by

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2, \\ \frac{1}{|\mathbb{S}^{n-1}|(n-2)|x|^{n-2}}, & n \geq 3. \end{cases}$$

For any real number $p > 1$, we let p' denote its conjugate index, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $p = 1$, we set $p' = \infty$.

We recall the definition of curl operator. When $n = 2$, curl operator is defined by

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ v_1 & v_2 \end{vmatrix} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \text{ for any } \mathbf{v} = (v_1, v_2) \in \mathcal{D}'(\Omega, \mathbb{R}^2).$$

When $n = 3$, curl operator is defined by

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \text{ for any } \mathbf{v} = (v_1, v_2, v_3) \in \mathcal{D}'(\Omega, \mathbb{R}^3).$$

There is another definition of curl operator, which is defined as a matrix of order $n \geq 2$. We denote the element of the matrix $\operatorname{CURL} \mathbf{v}$ by

$$\operatorname{CURL}_{ij} \mathbf{v} = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}, \quad i, j = 1, 2, \dots, n, \text{ for any } \mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathcal{D}'(\Omega, \mathbb{R}^n).$$

For a matrix $\mathbf{A} = (a_{ij}) \in \mathcal{D}'(\Omega, \mathbb{R}^{n^2})$, where $i, j = 1, 2, \dots, n$, we define its divergence by

$$\operatorname{div} \mathbf{A} = \left(\sum_{j=1}^n \frac{\partial a_{1j}}{\partial x_j}, \sum_{j=1}^n \frac{\partial a_{2j}}{\partial x_j}, \dots, \sum_{j=1}^n \frac{\partial a_{nj}}{\partial x_j} \right).$$

For any $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathcal{D}'(\Omega, \mathbb{R}^n)$, it holds that

$$\operatorname{div} \operatorname{CURL} \mathbf{v} = \Delta \mathbf{v} - \nabla(\operatorname{div} \mathbf{v}). \quad (2.1)$$

Throughout this paper, bold typeface will indicate vector or matrix quantities; normal typeface will be used for vector and matrix components and for scalars. To simplify the notations, we write $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\dot{W}^{s,p}}$ instead of $\|\cdot\|_{L^p(\mathbb{R}^n)}$ and $\|\cdot\|_{\dot{W}^{s,p}(\mathbb{R}^n)}$, respectively.

3. PROOFS OF THEOREM 1.4 AND THEOREM 1.5

We supply two tips to help to verify that whether the kernel $K(x)$ satisfies the conditions in (1.5) or not.

Proposition 3.1. *Let $0 < \lambda < n$. We have*

- (1) *If $|\nabla K(x)| \leq C|x|^{\lambda-n-1}$ for $|x| \neq 0$, then $K(x)$ satisfies the second condition (ii) in (1.5).*
- (2) *If the kernel has the form $K(x) = \frac{\omega(x)}{|x|^{n-\lambda}}$, where $\omega(x)$ is a Lipschitz continuous function on the unit sphere \mathbb{S}^{n-1} and satisfies the homogeneous condition of degree 0, i.e., $\omega(tx) = \omega(x)$ holds for any $t > 0$ and every $x \in \mathbb{R}^n$, then $K(x)$ satisfies the conditions in (1.5).*

Proof. Let $|y| \leq \frac{|x|}{2}$.

(1) By differential mean value theorem, there exists a $\theta \in (0, 1)$ such that

$$|K(x - y) - K(x)| = |y \cdot (\nabla K)(x - \theta y)|.$$

Since $|x - \theta y| \geq |x| - \theta|y| \geq |x| - |y| \geq \frac{|x|}{2}$, we have

$$|K(x - y) - K(x)| \leq C \frac{|y|}{|x - \theta y|^{n+1-\lambda}} \leq C \frac{|y|}{|x|^{n+1-\lambda}}.$$

(2) Obviously, we only need to verify the inequality

$$\left| \frac{\omega(x - y)}{|x - y|^{n-\lambda}} - \frac{\omega(x)}{|x|^{n-\lambda}} \right| \leq C \frac{|y|}{|x|^{n+1-\lambda}}. \quad (3.1)$$

Since

$$\left| \frac{1}{|x - y|^{n-\lambda}} - \frac{1}{|x|^{n-\lambda}} \right| \leq C \frac{|y|}{|x|^{n+1-\lambda}}$$

and

$$\left| \frac{x - y}{|x - y|} - \frac{x}{|x|} \right| \leq 2 \frac{|y|}{|x|},$$

we have

$$\begin{aligned} \left| \frac{\omega(x - y)}{|x - y|^{n-\lambda}} - \frac{\omega(x)}{|x|^{n-\lambda}} \right| &= \left| \frac{\omega(x - y)}{|x - y|^{n-\lambda}} - \frac{\omega(x - y)}{|x|^{n-\lambda}} + \frac{\omega(x - y)}{|x|^{n-\lambda}} - \frac{\omega(x)}{|x|^{n-\lambda}} \right| \\ &\leq \left| \omega\left(\frac{x - y}{|x - y|}\right) \right| \left| \frac{1}{|x - y|^{n-\lambda}} - \frac{1}{|x|^{n-\lambda}} \right| + |\omega(x - y) - \omega(x)| \frac{1}{|x|^{n-\lambda}} \\ &\leq \|\omega\|_{L^\infty(\mathbb{S}^{n-1})} \frac{C|y|}{|x|^{n+1-\lambda}} + \left| \omega\left(\frac{x - y}{|x - y|}\right) - \omega\left(\frac{x}{|x|}\right) \right| \frac{1}{|x|^{n-\lambda}} \\ &\leq \|\omega\|_{L^\infty(\mathbb{S}^{n-1})} \frac{C|y|}{|x|^{n+1-\lambda}} + \text{Lip}(\omega) \left| \frac{x - y}{|x - y|} - \frac{x}{|x|} \right| \frac{1}{|x|^{n-\lambda}} \\ &\leq C(\|\omega\|_{L^\infty(\mathbb{S}^{n-1})} + \text{Lip}(\omega)) \frac{|y|}{|x|^{n+1-\lambda}}, \end{aligned}$$

where $\text{Lip}(\omega)$ is the Lipschitz constant of $\omega(x)$ on \mathbb{S}^{n-1} . \square

Lemma 3.2. *Let $0 < \lambda < n$ and $0 < \alpha < 1$. Suppose that $K(x)$ satisfies the conditions in (1.5). Then for any $|x| \neq 0$, we have*

$$\int_{\mathbb{R}^n} \frac{|K(x) - K(y)|}{|x - y|^{n+\alpha}} dy \leq C|x|^{\lambda-n-\alpha}.$$

Hence we can define $(-\Delta)^{\frac{\alpha}{2}} K$ as

$$(-\Delta)^{\frac{\alpha}{2}} K(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{K(x) - K(y)}{|x - y|^{n+\alpha}} dy$$

and we have the estimate

$$|(-\Delta)^{\frac{\alpha}{2}} K(x)| \leq C|x|^{\lambda-n-\alpha} \text{ for any } |x| \neq 0.$$

Proof. It holds that

$$\int_{\mathbb{R}^n} \frac{|K(x) - K(y)|}{|x - y|^{n+\alpha}} dy \leq \int_{|x-y| \geq \frac{|x|}{2}} \frac{|K(x)| + |K(y)|}{|x - y|^{n+\alpha}} dy + \int_{|x-y| \leq \frac{|x|}{2}} \frac{|K(x) - K(y)|}{|x - y|^{n+\alpha}} dy. \quad (3.2)$$

By the condition (i) in (1.5), we obtain

$$\int_{|x-y| \geq \frac{|x|}{2}} \frac{|K(x)|}{|x - y|^{n+\alpha}} dy \leq C|x|^{\lambda-n} \int_{|x-y| \geq \frac{|x|}{2}} \frac{1}{|x - y|^{n+\alpha}} dy = C|x|^{\lambda-n-\alpha}. \quad (3.3)$$

By the condition (i) in (1.5), we have

$$\begin{aligned} \int_{|x-y| \geq \frac{|x|}{2}} \frac{|K(y)|}{|x - y|^{n+\alpha}} dy &= \int_{2|x| \geq |x-y| \geq \frac{|x|}{2}} \frac{|K(y)|}{|x - y|^{n+\alpha}} dy + \int_{|x-y| \geq 2|x|} \frac{|K(y)|}{|x - y|^{n+\alpha}} dy \\ &\leq C \left(\int_{2|x| \geq |x-y| \geq \frac{|x|}{2}} \frac{|y|^{\lambda-n}}{|x - y|^{n+\alpha}} dy + \int_{|x-y| \geq 2|x|} \frac{|y|^{\lambda-n}}{|x - y|^{n+\alpha}} dy \right) \\ &\leq C \left(|x|^{-n-\alpha} \int_{|y| \leq 3|x|} |y|^{\lambda-n} dy + |x|^{\lambda-n} \int_{|x-y| \geq 2|x|} \frac{1}{|x - y|^{n+\alpha}} dy \right) \\ &= C|x|^{\lambda-n-\alpha}. \end{aligned} \quad (3.4)$$

Thanks to the condition (ii) in (1.5), if $|x - y| \leq \frac{|x|}{2}$ then

$$|K(x) - K(y)| = |K(x - (x - y)) - K(x)| \leq C \frac{|x - y|}{|x|^{n+1-\lambda}}.$$

Hence

$$\begin{aligned} \int_{|x-y| \leq \frac{|x|}{2}} \frac{|K(x) - K(y)|}{|x - y|^{n+\alpha}} dy &\leq C|x|^{\lambda-n-1} \int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x - y|^{n+\alpha-1}} dy \\ &= C|x|^{\lambda-n-1} \cdot |x|^{1-\alpha} = C|x|^{\lambda-n-\alpha}. \end{aligned} \quad (3.5)$$

Therefore, this lemma completes by combining the above four inequalities. \square

Lemma 3.3 (cf. [23, p. 118]). *Let $\alpha, \beta > 0$ and $\alpha + \beta < n$. Then we have*

$$\int_{\mathbb{R}^n} \frac{|y|^{\alpha-n}}{|x - y|^{n-\beta}} dy = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha + \beta)} |x|^{\alpha+\beta-n}.$$

Remark 3.4. *Lemma 3.3 implies $(-\Delta)^{-\frac{\beta}{2}}|x|^{\alpha-n} = C(\alpha, \beta)|x|^{\alpha+\beta-n}$. More generally, if $0 < \alpha < n$, $\beta < 2$ and $0 < \alpha - \beta < n$, then $(-\Delta)^{\frac{\beta}{2}}|x|^{\alpha-n} = C(\alpha, \beta)|x|^{\alpha-\beta-n}$, cf. [13, p. 931]. This equality is an easy consequence of a scaling transform and an orthogonal transform, except for the value of the constant.*

Lemma 3.5. *Let $0 < \alpha < 1$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\text{supp } \varphi \subseteq B_R(0)$, where $R > 0$. Then we have*

$$(1) \quad \left| (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) \right| \leq C \|\varphi\|_{L^\infty} R^\alpha \text{ for any } x \in \mathbb{R}^n \text{ and } \left| (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) \right| \leq C \|\varphi\|_{L^1} |x|^{\alpha-n} \text{ for any } |x| \geq 2R.$$

(2) It holds that

$$\int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+\alpha}} dy \leq C(\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty}) \text{ for any } x \in \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+\alpha}} dy \leq C\|\varphi\|_{L^1} |x|^{-\alpha-n} \text{ for any } |x| \geq 2R.$$

Consequently, $|(-\Delta)^{\frac{\alpha}{2}}\varphi(x)| \leq C(\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty})$ for any $x \in \mathbb{R}^n$ and

$$|(-\Delta)^{\frac{\alpha}{2}}\varphi(x)| \leq C\|\varphi\|_{L^1} |x|^{-\alpha-n} \text{ for any } |x| \geq 2R.$$

(3) For any $x, y \in \mathbb{R}^n$, it holds that

$$|(-\Delta)^{-\frac{\alpha}{2}}\varphi(x) - (-\Delta)^{-\frac{\alpha}{2}}\varphi(y)| \leq C\|\nabla\varphi\|_{L^\infty} |x - y|(R + |x| + |y|)^\alpha.$$

(4) For any $|x| \neq 0$, it holds that

$$\int_{\mathbb{R}^n} \frac{|(-\Delta)^{-\frac{\alpha}{2}}\varphi(x) - (-\Delta)^{-\frac{\alpha}{2}}\varphi(y)|}{|x - y|^{n+\alpha}} dy \leq C(\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty} (R + |x|)^\alpha R^{1-\alpha}). \quad (3.6)$$

Moreover, for any $|x| \geq 2R$, it holds that

$$\int_{\mathbb{R}^n} \frac{|(-\Delta)^{-\frac{\alpha}{2}}\varphi(x) - (-\Delta)^{-\frac{\alpha}{2}}\varphi(y)|}{|x - y|^{n+\alpha}} dy \leq C\|\varphi\|_{L^1} |x|^{-n}. \quad (3.7)$$

Proof. (1) It holds that

$$\begin{aligned} |(-\Delta)^{-\frac{\alpha}{2}}\varphi(x)| &\leq C\|\varphi\|_{L^\infty} \int_{B_R(0)} \frac{1}{|x - y|^{n-\alpha}} dy \\ &\leq C\|\varphi\|_{L^\infty} \int_{B_R(0)} \frac{1}{|y|^{n-\alpha}} dy = C\|\varphi\|_{L^\infty} R^\alpha. \end{aligned}$$

For any $|x| \geq 2R$, we have

$$|(-\Delta)^{-\frac{\alpha}{2}}\varphi(x)| \leq C|x|^{\alpha-n} \int_{B_R(0)} |\varphi(y)| dy = C\|\varphi\|_{L^1} |x|^{\alpha-n}.$$

(2) We have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+\alpha}} dy &= \int_{|x-y| \leq 1} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+\alpha}} dy + \int_{|x-y| \geq 1} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+\alpha}} dy \\ &\leq \int_{|x-y| \leq 1} \frac{\|\nabla\varphi\|_{L^\infty} |x - y|}{|x - y|^{n+\alpha}} dy + \int_{|x-y| \geq 1} \frac{2\|\varphi\|_{L^\infty}}{|x - y|^{n+\alpha}} dy \\ &= C(\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^\infty}). \end{aligned}$$

For any $|x| \geq 2R$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+\alpha}} dy &= \int_{B_R(0)} \frac{|\varphi(y)|}{|x - y|^{n+\alpha}} dy \\ &\leq C\|\varphi\|_{L^1} |x|^{-\alpha-n}. \end{aligned}$$

(3) We can write

$$\begin{aligned}\varphi(x-z) - \varphi(y-z) &= \int_0^1 \frac{d}{dt} \varphi(tx + (1-t)y - z) dt \\ &= \int_0^1 (\nabla \varphi)(tx + (1-t)y - z) \cdot (x-y) dt.\end{aligned}$$

Hence we obtain

$$\begin{aligned}|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)| &\leq C \int_{\mathbb{R}^n} \frac{|\varphi(x-z) - \varphi(y-z)|}{|z|^{n-\alpha}} dz \\ &\leq C|x-y| \int_0^1 \int_{\mathbb{R}^n} \frac{|\nabla \varphi|(tx + (1-t)y - z)}{|z|^{n-\alpha}} dz dt \\ &\leq C|x-y| \|\nabla \varphi\|_{L^\infty} \int_{|z| \leq R+|x|+|y|} \frac{1}{|z|^{n-\alpha}} dz \\ &\leq C \|\nabla \varphi\|_{L^\infty} |x-y| (R+|x|+|y|)^\alpha.\end{aligned}$$

(4) Similar to the proof of the second item, we write

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy &= \int_{|x-y| \geq R} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)| + |(-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy \\ &\quad + \int_{|x-y| \leq R} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy\end{aligned}\tag{3.8}$$

By the first estimate in the first item, we get

$$\begin{aligned}\int_{|x-y| \geq R} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)| + |(-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy &\leq C \|\varphi\|_{L^\infty} R^\alpha \int_{|x-y| \geq R} \frac{1}{|x-y|^{n+\alpha}} dy \\ &= C \|\varphi\|_{L^\infty}.\end{aligned}\tag{3.9}$$

By the estimate in the third item, we get

$$\begin{aligned}\int_{|x-y| \leq R} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy &\leq C \|\nabla \varphi\|_{L^\infty} \int_{|x-y| \leq R} \frac{(R+|x|+|y|)^\alpha}{|x-y|^{n+\alpha-1}} dy \\ &\leq C \|\nabla \varphi\|_{L^\infty} (R+|x|)^\alpha R^{1-\alpha}.\end{aligned}\tag{3.10}$$

Combining the above three inequalities, we obtain the estimate (3.6).

For any $|x| \geq 2R$, by Fubini's theorem and Lemma 3.2, we get

$$\int_{\mathbb{R}^n} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy$$

$$\begin{aligned}
&\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| |\varphi(z)| dz \frac{1}{|x-y|^{n+\alpha}} dy \\
&\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| \frac{1}{|x-y|^{n+\alpha}} dy |\varphi(z)| dz \\
&= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right| \frac{1}{|(x-z)-y|^{n+\alpha}} dy |\varphi(z)| dz \\
&\leq C \int_{\mathbb{R}^n} \frac{1}{|x-z|^n} |\varphi(z)| dz = C \int_{B_R(0)} \frac{1}{|x-z|^n} |\varphi(z)| dz \leq C \|\varphi\|_{L^1} |x|^{-n}.
\end{aligned} \tag{3.11}$$

□

By Remark 3.4, we get $(-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}|x|^{\lambda-n} = C_1(-\Delta)^{-\frac{\alpha}{2}}|x|^{\lambda-n-\alpha} = C_1C_2|x|^{\lambda-n}$, where $0 < \alpha < \lambda$. Moreover, $C_1C_2 = 1$, cf. [13, p. 931]. Next lemma indicates that $K(x)$ also has this similar property.

Lemma 3.6. *Let $0 < \lambda < n$ and $\alpha = \min\{\frac{\lambda}{n}, \frac{n-\lambda}{2}\}$. Suppose that $K(x)$ satisfies the conditions in (1.5) and $(-\Delta)^{\frac{\alpha}{2}}K(x)$ is defined as Lemma 3.2. Then we have*

$$(-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K(x) = K(x) \text{ for any } |x| \neq 0.$$

Proof. The parameter $\alpha > 0$ is to be determined. We first show that $(-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K$ is well-defined. In fact, by Lemma 3.2 and 3.3, we get

$$\begin{aligned}
|(-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K(x)| &\leq (-\Delta)^{-\frac{\alpha}{2}}|(-\Delta)^{\frac{\alpha}{2}}K(x)| \\
&\leq C(-\Delta)^{-\frac{\alpha}{2}}|x|^{\lambda-n-\alpha} = C|x|^{\lambda-n},
\end{aligned}$$

here we require $\alpha < \lambda$ and $\alpha < 1$.

Moreover, for any $0 < \tau < \alpha$, $(-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K$ is a function of $C^\tau(\mathbb{R}^n \setminus \{0\})$. Let $x_0 \in \mathbb{R}^n \setminus \{0\}$ and x be a point near x_0 . Thanks to the inequality (cf. [17, p. 261])

$$|a^{-m} - b^{-m}| \leq m \left(\frac{1-\gamma}{m+\gamma} \right)^{1-\gamma} |a-b|^\gamma (a^{-m-\gamma} + b^{-m-\gamma}),$$

where $a, b > 0$, $m > 0$ and $\gamma \in (0, 1)$, and using Lemma 3.2 and 3.3, we obtain

$$\begin{aligned}
&|(-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K(x) - (-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K(x_0)| \\
&= C \left| \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right) (-\Delta)^{\frac{\alpha}{2}}K(y) dy \right| \\
&\leq C \int_{\mathbb{R}^n} ||x-y| - |x_0-y||^\tau (|x-y|^{\alpha-n-\tau} + |x_0-y|^{\alpha-n-\tau}) |y|^{\lambda-n-\alpha} dy \\
&\leq C|x-x_0|^\tau \int_{\mathbb{R}^n} (|x-y|^{\alpha-n-\tau} + |x_0-y|^{\alpha-n-\tau}) |y|^{\lambda-n-\alpha} dy \\
&= C|x-x_0|^\tau (|x|^{\lambda-n-\tau} + |x_0|^{\lambda-n-\tau}).
\end{aligned}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\text{supp } \varphi \subseteq B_R(0)$. By Fubini's theorem, we have

$$\int_{\mathbb{R}^n} ((-\Delta)^{-\frac{\alpha}{2}}(-\Delta)^{\frac{\alpha}{2}}K(x)) \varphi(x) dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}}K(x) (-\Delta)^{-\frac{\alpha}{2}}\varphi(x) dx. \tag{3.12}$$

By Lemma 3.2 and 3.5, we get

$$\left| \int_{|x-y| \geq \varepsilon} \frac{K(x) - K(y)}{|x-y|^{n+\alpha}} dy (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) \right| \leq C|x|^{\lambda-n-\alpha} |(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)| \in L^1(\mathbb{R}^n).$$

Hence, by Lebesgue's dominated convergence theorem, we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} C_{n,\alpha} \int_{|x-y| \geq \varepsilon} \frac{K(x) - K(y)}{|x-y|^{n+\alpha}} dy (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} K (-\Delta)^{-\frac{\alpha}{2}} \varphi dx. \quad (3.13)$$

By a simple calculation, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{|x-y| \geq \varepsilon} \frac{K(x) - K(y)}{|x-y|^{n+\alpha}} dy (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx \\ &= \frac{\varepsilon^{-\alpha}}{\alpha} |\mathbb{S}^{n-1}| \int_{\mathbb{R}^n} K(x) (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx - \int_{\mathbb{R}^n} \int_{|x-y| \geq \varepsilon} \frac{K(y)}{|x-y|^{n+\alpha}} dy (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx, \end{aligned}$$

here we require $\alpha < n - \lambda$ to guarantee $K(x) (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) \in L^1(\mathbb{R}^n)$. When $\varepsilon \geq \frac{|x|}{2}$, by the inequality (3.4), we have

$$\int_{|x-y| \geq \varepsilon} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy \leq \int_{|x-y| \geq \frac{|x|}{2}} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy \leq C|x|^{\lambda-n-\alpha}.$$

When $\varepsilon < \frac{|x|}{2}$, we have

$$\begin{aligned} \int_{|x-y| \geq \varepsilon} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy &\leq \int_{|x-y| \geq \frac{|x|}{2}} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy + \int_{\varepsilon \leq |x-y| \leq \frac{|x|}{2}} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy \\ &\leq C(|x|^{\lambda-n-\alpha} + \varepsilon^{-\alpha} |x|^{\lambda-n}). \end{aligned}$$

In conclusion, we obtain

$$\int_{|x-y| \geq \varepsilon} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy \leq C(|x|^{\lambda-n-\alpha} + \varepsilon^{-\alpha} |x|^{\lambda-n}).$$

Hence by Lemma 3.5, we have

$$\begin{aligned} & \int_{|x-y| \geq \varepsilon} \frac{|K(y)|}{|x-y|^{n+\alpha}} dy |(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)| \\ &\leq C(|x|^{\lambda-n-\alpha} + \varepsilon^{-\alpha} |x|^{\lambda-n}) |(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)| \in L^1(\mathbb{R}^n). \end{aligned}$$

By Fubini's theorem, then we obtain

$$\int_{\mathbb{R}^n} \int_{|x-y| \geq \varepsilon} \frac{K(y)}{|x-y|^{n+\alpha}} dy (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx = \int_{\mathbb{R}^n} \int_{|y-x| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)}{|y-x|^{n+\alpha}} dx K(y) dy.$$

Hence we have

$$C_{n,\alpha} \int_{\mathbb{R}^n} \int_{|x-y| \geq \varepsilon} \frac{K(x) - K(y)}{|x-y|^{n+\alpha}} dy (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx$$

$$\begin{aligned}
&= C_{n,\alpha} \left(\frac{\varepsilon^{-\alpha}}{\alpha} |\mathbb{S}^{n-1}| \int_{\mathbb{R}^n} K(x) (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx - \int_{\mathbb{R}^n} \int_{|y-x| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)}{|y-x|^{n+\alpha}} dx K(y) dy \right) \\
&= C_{n,\alpha} \left(\frac{\varepsilon^{-\alpha}}{\alpha} |\mathbb{S}^{n-1}| \int_{\mathbb{R}^n} K(y) (-\Delta)^{-\frac{\alpha}{2}} \varphi(y) dy - \int_{\mathbb{R}^n} \int_{|y-x| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(x)}{|y-x|^{n+\alpha}} dx K(y) dy \right) \\
&= C_{n,\alpha} \int_{\mathbb{R}^n} \int_{|y-x| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(y) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(x)}{|y-x|^{n+\alpha}} dx K(y) dy \\
&= C_{n,\alpha} \int_{\mathbb{R}^n} \int_{|x-y| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)}{|x-y|^{n+\alpha}} dy K(x) dx.
\end{aligned} \tag{3.14}$$

By Lemma 3.5, we obtain

$$\begin{aligned}
&\left| \int_{|x-y| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)}{|x-y|^{n+\alpha}} dy K(x) \right| \\
&\leq \int_{\mathbb{R}^n} \frac{|(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)|}{|x-y|^{n+\alpha}} dy |K(x)| \in L^1(\mathbb{R}^n).
\end{aligned}$$

Then by Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} C_{n,\alpha} \int_{|x-y| \geq \varepsilon} \frac{(-\Delta)^{-\frac{\alpha}{2}} \varphi(x) - (-\Delta)^{-\frac{\alpha}{2}} \varphi(y)}{|x-y|^{n+\alpha}} dy K(x) dx \\
&= \int_{\mathbb{R}^n} ((-\Delta)^{\frac{\alpha}{2}} (-\Delta)^{-\frac{\alpha}{2}} \varphi(x)) K(x) dx.
\end{aligned} \tag{3.15}$$

Combining the equalities (3.13), (3.14) and (3.15), we obtain

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} K(x) (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) dx = \int_{\mathbb{R}^n} K(x) ((-\Delta)^{\frac{\alpha}{2}} (-\Delta)^{-\frac{\alpha}{2}} \varphi(x)) dx. \tag{3.16}$$

Hence, by the equality (3.12), we get

$$\int_{\mathbb{R}^n} ((-\Delta)^{-\frac{\alpha}{2}} (-\Delta)^{\frac{\alpha}{2}} K(x)) \varphi(x) dx = \int_{\mathbb{R}^n} K(x) \varphi(x) dx,$$

here we use a result that $(-\Delta)^{\frac{\alpha}{2}} (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) = \varphi(x)$ a.e. $x \in \mathbb{R}^n$ for any $\varphi \in C_c^\infty(\mathbb{R}^n)$. Actually, for any $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$, it is similar to the equality (3.16) but much easier to get the following equality:

$$\int_{\mathbb{R}^n} ((-\Delta)^{\frac{\alpha}{2}} (-\Delta)^{-\frac{\alpha}{2}} \varphi(x)) \psi(x) dx = \int_{\mathbb{R}^n} (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) (-\Delta)^{\frac{\alpha}{2}} \psi(x) dx.$$

By Lemma 3.5, we have $(-\Delta)^{-\frac{\alpha}{2}} \varphi \in L^2(\mathbb{R}^n)$ and $(-\Delta)^{\frac{\alpha}{2}} \psi \in L^2(\mathbb{R}^n)$, here we require $\alpha < \frac{n}{2}$. Taking into consideration the conditions on α , we choose $\alpha = \min\{\frac{\lambda}{n}, \frac{n-\lambda}{2}\}$. By Plancherel theorem and [12, Proposition 3.3], we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} (-\Delta)^{-\frac{\alpha}{2}} \varphi(x) (-\Delta)^{\frac{\alpha}{2}} \psi(x) dx &= \int_{\mathbb{R}^n} |\xi|^{-\alpha} \hat{\varphi} \overline{|\xi|^\alpha \hat{\psi}} d\xi \\
&= \int_{\mathbb{R}^n} \hat{\varphi} \overline{\hat{\psi}} d\xi = \int_{\mathbb{R}^n} \varphi(x) \psi(x) dx,
\end{aligned}$$

which completes our proof. \square

Proof of Theorem 1.4. Step 1. First, we deal with the divergence-free vector field, i.e., $\operatorname{div} \mathbf{f} = 0$. For any $\lambda \in (0, n)$, set $\alpha = \min\{\frac{\lambda}{n}, \frac{n-\lambda}{2}\}$. By Lemma 3.6, Fubini's theorem and Theorem 1.1, we have

$$\begin{aligned} T_\lambda \mathbf{f}(x) &= \int_{\mathbb{R}^n} K(y) \mathbf{f}(x-y) dy \\ &= \int_{\mathbb{R}^n} ((-\Delta)^{-\frac{\alpha}{2}} (-\Delta)^{\frac{\alpha}{2}} K(y)) \mathbf{f}(x-y) dy \\ &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} K(y) (-\Delta)^{-\frac{\alpha}{2}} \mathbf{f}(x-y) dy. \end{aligned}$$

By Lemma 3.2, Theorem 1.1 and 1.3, we obtain

$$\begin{aligned} \|T_\lambda \mathbf{f}\|_{L^q} &\leq C \left\| \int_{\mathbb{R}^n} \frac{1}{|y|^{n-(\lambda-\alpha)}} |(-\Delta)^{-\frac{\alpha}{2}} \mathbf{f}(x-y)| dy \right\|_{L^q} \\ &= C \left\| \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-(\lambda-\alpha)}} |(-\Delta)^{-\frac{\alpha}{2}} \mathbf{f}(y)| dy \right\|_{L^q} \\ &\leq C \|(-\Delta)^{-\frac{\alpha}{2}} \mathbf{f}\|_{L^r} \leq C \|\mathbf{f}\|_{L^1}, \end{aligned}$$

where $\frac{1}{r} = 1 - \frac{\alpha}{n}$.

Step 2. Next, we turn to non-divergence-free vector field. We decompose \mathbf{f} as $\mathbf{f} = \mathbf{f} + \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} - \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}$, which is called the Leray decomposition of \mathbf{f} . $\mathbf{f} + \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}$ is the divergence-free part while $\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}$ is the curl-free part. Since $\operatorname{div}(\mathbf{f} + \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}) = 0$, by Step 1, we obtain

$$\begin{aligned} \|T_\lambda(\mathbf{f} + \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f})\|_{L^q} &\leq C \|\mathbf{f} + \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1} \\ &\leq C(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}). \end{aligned} \quad (3.17)$$

On the other hand, for $1 \leq i < j \leq n$, set $(g_i, g_j) = (\frac{\partial}{\partial x_j}(-\Delta)^{-1} \operatorname{div} \mathbf{f}, -\frac{\partial}{\partial x_i}(-\Delta)^{-1} \operatorname{div} \mathbf{f})$, $g_k = 0$ if $k \neq i, j$. Then we have $\operatorname{div} \mathbf{g} = 0$. By Step 1, we obtain

$$\begin{aligned} \left\| T_\lambda \left(\frac{\partial}{\partial x_i} (-\Delta)^{-1} \operatorname{div} \mathbf{f} \right) \right\|_{L^q} + \left\| T_\lambda \left(\frac{\partial}{\partial x_j} (-\Delta)^{-1} \operatorname{div} \mathbf{f} \right) \right\|_{L^q} &\leq 2 \|T_\lambda \mathbf{g}\|_{L^q} \leq C \|\mathbf{g}\|_{L^1} \\ &\leq C \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}. \end{aligned}$$

Hence, we have

$$\|T_\lambda(\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f})\|_{L^q} \leq C \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}. \quad (3.18)$$

Therefore, the inequality (1.6) follows from the inequalities (3.17) and (3.18). \square

By the way, using our idea, we can get an improvement of [2, Proposition 5.19], where they require that the kernel $K(x)$ is a $C^\infty(\mathbb{R}^n \setminus \{0\})$ function satisfying $|K(x)| \leq C|x|^{\lambda-n}$ and $|\nabla K(x)| \leq C|x|^{\lambda-n-1}$. In the following proposition, we weaken the smoothness of the kernels and expand the range of the kernels.

Proposition 3.7. *Let $0 < \lambda < n$, $\frac{1}{q} = 1 - \frac{\lambda}{n}$. Assume that $f \in \mathcal{H}^1(\mathbb{R}^n)$. Then*

$$\|T_\lambda f\|_{L^q} \leq C \|f\|_{\mathcal{H}^1}. \quad (3.19)$$

Moreover, when $n \geq 2$, we have a slightly strong version:

$$\|T_\lambda f\|_{L^q} \leq C \sum_{j=1}^n \|R_j f\|_{L^1}.$$

Proof. By the same idea of the proof of Theorem 1.4, we have

$$\|T_\lambda f\|_{L^q} \leq C \|(-\Delta)^{-\frac{\alpha}{2}} f\|_{L^r},$$

where $\frac{1}{r} = 1 - \frac{\alpha}{n}$ and $\alpha = \min\{\frac{\lambda}{n}, \frac{n-\lambda}{2}\}$. Thus we complete our proof by using the following inequality (see [25, Theorem H])

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_{L^r} \leq C \|f\|_{\mathcal{H}^1}.$$

When $n \geq 2$, by [21, Theorem A] and a standard approximation argument, the above inequality can be replaced by

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_{L^r} \leq C \sum_{j=1}^n \|R_j f\|_{L^1}.$$

□

Let $n \geq 2$ and $0 < \lambda < n$. We point out that if $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ then $\operatorname{div} \mathbf{f} \in \dot{W}^{-\lambda-1, \frac{n}{n-\lambda}}(\mathbb{R}^n)$. We verify this fact in three cases. When $\lambda = 1$, by Gagliardo-Nirenberg inequality, we have $(-\Delta)^{-1} \operatorname{div} \mathbf{f} \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$. When $1 < \lambda < n$, since

$$(-\Delta)^{-\frac{\lambda+1}{2}} \operatorname{div} \mathbf{f} = (-\Delta)^{-\frac{\lambda-1}{2}} (-\Delta)^{-1} \operatorname{div} \mathbf{f},$$

by Theorem 1.1, we get $(-\Delta)^{-\frac{\lambda+1}{2}} \operatorname{div} \mathbf{f} \in L^{\frac{n}{n-\lambda}}(\mathbb{R}^n)$. When $0 < \lambda < 1$, since

$$\nabla(-\Delta)^{-\frac{1-\lambda}{2}} (-\Delta)^{-\frac{\lambda+1}{2}} \operatorname{div} \mathbf{f} = \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n),$$

by [21, Theorem A'], we obtain $(-\Delta)^{-\frac{\lambda+1}{2}} \operatorname{div} \mathbf{f} \in L^{\frac{n}{n-\lambda}}(\mathbb{R}^n)$. We can also illustrate this fact as follows. By Theorem 1.4 with $K(x) = |x|^{\lambda-n}$, we get $\mathbf{f} \in \dot{W}^{-\lambda, \frac{n}{n-\lambda}}(\mathbb{R}^n, \mathbb{R}^n)$, then it follows that $\operatorname{div} \mathbf{f} \in \dot{W}^{-\lambda-1, \frac{n}{n-\lambda}}(\mathbb{R}^n)$. Inspired by inequality (1.2) and the above fact, naturally, we consider extending Theorem 1.3 and Theorem 1.4 to the vector field, whose divergence is in negative Sobolev spaces, and raise the following question.

Problem 3.8. *Let $n \geq 2$, $0 < \lambda < n$. Is it true that*

$$\|\mathbf{f}\|_{\dot{W}^{-\lambda, \frac{n}{n-\lambda}}} \leq C(\|\mathbf{f}\|_{L^1} + \|\operatorname{div} \mathbf{f}\|_{\dot{W}^{-\lambda-1, \frac{n}{n-\lambda}}})? \quad (3.20)$$

This question is closely related to a question raised in [5, Open Problem 2]. Maz'ya [19] solved a special case of the open problem. Recently, Bousquet, Mironescu and Russ [7] have made some progress on this open problem.

Remark 3.9. Applying inequality (1.3) to \mathbf{g} in the proof of Theorem 1.4, we can get

$$\left\| \frac{\nabla(-\Delta)^{-1} \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}}{|x|^\beta} \right\|_{L^q} \leq C \left\| \nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} \right\|_{L^1},$$

where $1 \leq q < n'$, $\beta = 1 - n(1 - \frac{1}{q})$. Using this inequality, we can derive inequality (1.4) for $1 \leq q < n'$ from inequality (1.3) by the same method as the proof of Theorem 1.4.

Proposition 3.10. Let $1 \leq p < \infty$. Then $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n) = \mathbf{H}_0^p(\operatorname{div}, \mathbb{R}^n)$.

Proof. It suffices to show that $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is dense in $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$. Assume that $\mathbf{v} \in \mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$. Set $\mathbf{v}^\varepsilon = \eta_\varepsilon * \mathbf{v}$, where η is the standard mollifier. We have that $\mathbf{v}^\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathbf{v}^\varepsilon \rightarrow \mathbf{v}$ in $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$ as $\varepsilon \rightarrow 0$. Let $\zeta \in C_c^\infty(B_2(0))$ be a cut-off function such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ in $B_1(0)$. Set $\mathbf{v}_k^\varepsilon(x) = \mathbf{v}^\varepsilon(x) \zeta(\frac{x}{k})$, then $\mathbf{v}_k^\varepsilon \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. By the definition of divergence operator, we get

$$\operatorname{div} \mathbf{v}_k^\varepsilon(x) = \operatorname{div} \mathbf{v}^\varepsilon(x) \zeta(\frac{x}{k}) + \frac{1}{k} \mathbf{v}^\varepsilon(x) \cdot (\nabla \zeta)(\frac{x}{k}).$$

Therefore, for any fixed ε , we have

$$\|\operatorname{div} \mathbf{v}_k^\varepsilon - \operatorname{div} \mathbf{v}^\varepsilon\|_{L^p}^p \leq C \left(\int_{|x|>k} |\operatorname{div} \mathbf{v}^\varepsilon|^p dx + \frac{1}{k} \|\mathbf{v}^\varepsilon\|_{L^p}^p \right) \rightarrow 0 \text{ as } k \rightarrow \infty$$

and

$$\|\mathbf{v}_k^\varepsilon - \mathbf{v}^\varepsilon\|_{L^p}^p \leq \int_{|x|>k} |\mathbf{v}^\varepsilon|^p dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

By the above two inequalities and $\mathbf{v}^\varepsilon \rightarrow \mathbf{v}$ in $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$ as $\varepsilon \rightarrow 0$, we know that for any $\varepsilon > 0$, there exists $k = k(\varepsilon)$ such that $\mathbf{v}_k^\varepsilon \rightarrow \mathbf{v}$ in $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$ as $\varepsilon \rightarrow 0$. \square

Remark 3.11. Let $1 \leq p < \infty$. Then the set $\{\mathbf{v} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) : \operatorname{div} \mathbf{v} = 0\}$ is dense in $\mathbf{H}^p(\operatorname{div}, \mathbb{R}^n)$, cf. [1, p. 3156].

Since $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is dense in $\mathbf{H}^1(\operatorname{div}, \mathbb{R}^n)$, by the divergence theorem and an approximation argument, we obtain the following corollary.

Corollary 3.12. For any $\mathbf{v} \in \mathbf{H}^1(\operatorname{div}, \mathbb{R}^n)$, it holds that

$$\int_{\mathbb{R}^n} \operatorname{div} \mathbf{v} dx = 0.$$

Let $\rho_0 \in C_c^\infty(\mathbb{R}^+)$ be a cut-off function such that $0 \leq \rho_0 \leq 1$ and

$$\rho_0(t) = \begin{cases} 1, & t \leq \frac{1}{4}, \\ 0, & t \geq \frac{1}{2}. \end{cases}$$

We denote $\rho(y, x) = \rho_0(\frac{|y|}{|x|})$ for $(y, x) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. We extract a lemma from the proof of the main theorem in [6], but in a simple case.

Lemma 3.13. Let $n \geq 2$, $\mathbf{f} \in L_{loc}^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{div} \mathbf{f} = 0$. Then we have

$$\left| \int_{\mathbb{R}^n} \rho(y, x) \mathbf{f}(y) dy \right| \leq C \int_{|y| \leq \frac{|x|}{2}} \frac{|y|}{|x|} |\mathbf{f}(y)| dy.$$

Proof. For any $|x| \neq 0$, we have $y_i \rho(y, x) \mathbf{f}(y) \in \mathbf{H}^1(\operatorname{div}, \mathbb{R}^n)$ and

$$\operatorname{div}(y_i \rho(y, x) \mathbf{f}(y)) = \nabla_y(y_i \rho(y, x)) \cdot \mathbf{f}(y) + y_i \rho(y, x) \operatorname{div} \mathbf{f} = \nabla_y(y_i \rho(y, x)) \cdot \mathbf{f}(y).$$

By Corollary 3.12, we get

$$\int_{\mathbb{R}^n} \operatorname{div}(y_i \rho(y, x) \mathbf{f}(y)) dy = 0.$$

Thus

$$\int_{\mathbb{R}^n} \rho(y, x) f_i(y) + \frac{y_i}{|y||x|} \rho'_0\left(\frac{|y|}{|x|}\right) \sum_{j=1}^n y_j f_j(y) dy = 0.$$

So we get

$$\left| \int_{\mathbb{R}^n} \rho(y, x) f_i(y) dy \right| \leq C \int_{|y| \leq \frac{|x|}{2}} \frac{|y|}{|x|} |\mathbf{f}(y)| dy.$$

Of course, we can also obtain this lemma in a similar way to [6]. First, we prove this lemma for smooth functions, then complete the proof by an approximation argument. \square

Proof of Theorem 1.5. In view of the proof of Theorem 1.4, we claim that Theorem 1.5 is equivalent to the following statement.

Let $n \geq 2$, $0 < \lambda < n$, $\alpha < 1$, $\beta < \frac{n}{q}$, $\alpha + \beta > 0$, $\frac{1}{q} = 1 + \frac{\alpha + \beta - \lambda}{n}$ and $\operatorname{div} \mathbf{f} = 0$. If $1 \leq q < \infty$, then

$$\| |x|^{-\beta} T_\lambda \mathbf{f} \|_{L^q} \leq C \| |x|^\alpha \mathbf{f} \|_{L^1}. \quad (3.21)$$

Hence, we only need to prove the case of $\operatorname{div} \mathbf{f} = 0$. The benefit of this observation is that there is no need to deal with the term $\int_{\mathbb{R}^n} y_i \rho(y, x) \operatorname{div} \mathbf{f}(y) dy$ in Lemma 3.13 for \mathbf{f} is a divergence-free vector field, which is different from [6].

We begin to prove the inequality (3.21).

Step 1. We write $T_\lambda \mathbf{f}(x) = J_1(x) + J_2(x)$, where

$$J_1(x) = \int_{\mathbb{R}^n} \rho(y, x) K(x - y) \mathbf{f}(y) dy, \quad J_2(x) = \int_{\mathbb{R}^n} (1 - \rho(y, x)) K(x - y) \mathbf{f}(y) dy.$$

By the condition (i) in (1.5) and generalized Minkowski's inequality, we have

$$\begin{aligned} \| |x|^{-\beta} J_2(x) \|_{L^q} &\leq C \left\| |x|^{-\beta} \int_{|y| \geq \frac{|x|}{4}} \frac{|\mathbf{f}(y)|}{|x - y|^{n-\lambda}} dy \right\|_{L^q} \\ &\leq C \int_{\mathbb{R}^n} \left(\int_{|x| \leq 4|y|} \frac{|x|^{-\beta q}}{|x - y|^{(n-\lambda)q}} dx \right)^{\frac{1}{q}} |\mathbf{f}(y)| dy. \end{aligned}$$

Since

$$\begin{aligned} \int_{|x| \leq 4|y|} \frac{|x|^{-\beta q}}{|x - y|^{(n-\lambda)q}} dx &= \int_{|x| \leq \frac{|y|}{2}} \frac{|x|^{-\beta q}}{|x - y|^{(n-\lambda)q}} dx + \int_{\frac{|y|}{2} \leq |x| \leq 4|y|} \frac{|x|^{-\beta q}}{|x - y|^{(n-\lambda)q}} dx \\ &\leq C (|y|^{(\lambda-n)q} \int_{|x| \leq \frac{|y|}{2}} |x|^{-\beta q} dx + |y|^{-\beta q} \int_{|x-y| \leq 5|y|} |x - y|^{(\lambda-n)q} dx) \\ &= C |y|^{n-\beta q + (\lambda-n)q} = C |y|^{\alpha q}, \end{aligned}$$

here we require $n - \beta q > 0$ and $n + (\lambda - n)q > 0$, then we get

$$\| |x|^{-\beta} J_2(x) \|_{L^q} \leq C \int_{\mathbb{R}^n} |y|^\alpha |\mathbf{f}(y)| dy. \quad (3.22)$$

Step 2. We write $J_1(x) = J_{11}(x) + J_{12}(x)$, where

$$J_{11}(x) = \int_{\mathbb{R}^n} \rho(y, x) (K(x - y) - K(x)) \mathbf{f}(y) dy, \quad J_{12}(x) = \int_{\mathbb{R}^n} \rho(y, x) K(x) \mathbf{f}(y) dy.$$

Thus by generalized Minkowski's inequality and the condition (ii) in (1.5), we obtain

$$\begin{aligned} \| |x|^{-\beta} J_{11}(x) \|_{L^q} &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (\rho(y, x) |x|^{-\beta} |K(x - y) - K(x)|)^q dx \right)^{\frac{1}{q}} |\mathbf{f}(y)| dy \\ &\leq C \int_{\mathbb{R}^n} \left(\int_{|x| \geq 2|y|} |x|^{(\lambda - n - 1 - \beta)q} dx \right)^{\frac{1}{q}} |y| |\mathbf{f}(y)| dy \\ &= C \int_{\mathbb{R}^n} (|y|^{(\alpha - 1)q})^{\frac{1}{q}} |y| |\mathbf{f}(y)| dy = C \int_{\mathbb{R}^n} |y|^\alpha |\mathbf{f}(y)| dy, \end{aligned} \quad (3.23)$$

here we require $n + (\lambda - n - 1 - \beta)q < 0$, i.e., $\alpha < 1$.

Step 3. At last we deal with the term $J_{12}(x)$. By the condition (i) in (1.5), we have

$$|J_{12}(x)| \leq C \frac{1}{|x|^{n-\lambda}} \left| \int_{\mathbb{R}^n} \rho(y, x) \mathbf{f}(y) dy \right|.$$

Due to Lemma 3.13 and generalized Minkowski's inequality, we get

$$\begin{aligned} \| |x|^{-\beta} J_{12}(x) \|_{L^q} &\leq C \left\| |x|^{\lambda - n - \beta} \int_{|y| \leq \frac{|x|}{2}} \frac{|y|}{|x|} |\mathbf{f}(y)| dy \right\|_{L^q} \\ &\leq C \int_{\mathbb{R}^n} \left(\int_{|x| \geq 2|y|} |x|^{(\lambda - n - \beta - 1)q} dx \right)^{\frac{1}{q}} |y| |\mathbf{f}(y)| dy \\ &= C \int_{\mathbb{R}^n} |y|^\alpha |\mathbf{f}(y)| dy, \end{aligned} \quad (3.24)$$

here we require $n + (\lambda - n - \beta - 1)q < 0$.

Combining the inequalities (3.22), (3.23) and (3.24), we obtain the result. \square

Remark 3.14. Let $\mathbf{u} = (-\Delta)^{-1} \mathbf{f}$. For $n \geq 3$, by Theorem 1.5 with $K(x) = |x|^{2-n}$, then we have

$$\left\| \frac{\mathbf{u}}{|x|^\beta} \right\|_{L^q} \leq C (\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}),$$

where $1 \leq q < \frac{n}{n-2}$, $\beta = 2 + n(\frac{1}{q} - 1)$. If we set $K(x) = \frac{x_j}{|x|^n}$ in Theorem 1.5, then we can see that our result generalizes the inequality (1.4) in a doubly weighted form.

4. APPLICATIONS

There are many proofs to Hardy inequality (cf. [10, p. 111]). Here we give a new proof of Hardy inequality by the theory of singular integrals. If $1 < p < n$, we can use Theorem 1.2 to prove Hardy inequality

$$\left\| \frac{u}{|x|} \right\|_{L^p} \leq C \|\nabla u\|_{L^p} \text{ for any } u \in C_c^\infty(\mathbb{R}^n).$$

But Theorem 1.2 doesn't work when $p = 1$. Our theorem make it possible to prove the case $p = 1$. At the same time, we can also give another approach to Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^q} \leq C \|\nabla u\|_{L^p} \text{ for any } u \in C_c^\infty(\mathbb{R}^n), \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{1}{n}, p \geq 1.$$

In fact, for any $u \in C_c^\infty(\mathbb{R}^n)$, we have the equality (see [14, Lemma 7.14])

$$u(x) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} \frac{(x-y) \cdot \nabla u(y)}{|x-y|^n} dy.$$

If $p > 1$, by Theorem 1.2, we have

$$\|u\|_{L^q} \leq \frac{1}{|\mathbb{S}^{n-1}|} \left\| \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \right\|_{L^q} \leq C \|\nabla u\|_{L^p}.$$

If $p > 1$, by Theorem 1.2, we have

$$\left\| \frac{u}{|x|} \right\|_{L^p} \leq \frac{1}{|\mathbb{S}^{n-1}|} \left\| \frac{1}{|x|} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy \right\|_{L^p} \leq C \|\nabla u\|_{L^p}.$$

In view of (2.1), for any function $\mathbf{f} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\| |x|^\alpha \mathbf{f} \|_{L^1} + \| |x|^\alpha \nabla (-\Delta)^{-1} \operatorname{div} \mathbf{f} \|_{L^1} \leq 2 (\| |x|^\alpha \mathbf{f} \|_{L^1} + \| |x|^\alpha \operatorname{div} (-\Delta)^{-1} \operatorname{CURL} \mathbf{f} \|_{L^1}).$$

Therefore, for divergence-free or curl-free smooth vector fields with compact support, the term $\| \nabla (-\Delta)^{-1} \operatorname{div} \mathbf{f} \|_{L^1}$ can be removed in the inequality (1.6) while the term $\| |x|^\alpha \nabla (-\Delta)^{-1} \operatorname{div} \mathbf{f} \|_{L^1}$ can be removed in the inequality (1.7).

If $p = 1$, since $\nabla (-\Delta)^{-1} \operatorname{div} \nabla u = -\nabla u$ or $\operatorname{CURL} \nabla u = \mathbf{0}$, setting $K(x) = \frac{x_j}{|x|^n}$ in Theorem 1.4, we get

$$\begin{aligned} \|u\|_{L^{n'}} &= \frac{1}{|\mathbb{S}^{n-1}|} \left\| \int_{\mathbb{R}^n} \frac{(x-y) \cdot \nabla u(y)}{|x-y|^n} dy \right\|_{L^{n'}} \\ &\leq \frac{1}{|\mathbb{S}^{n-1}|} \sum_{j=1}^n \left\| \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^n} \nabla u(y) dy \right\|_{L^{n'}} \\ &\leq C \|\nabla u\|_{L^1}. \end{aligned}$$

The above inequality can also be deduced from an inequality established by Bourgain and Brezis and Calderón-Zygmund inequality, see [26] or [27]. Setting $K(x) = \frac{x_j}{|x|^n}$ in Theorem

1.5, we get

$$\begin{aligned} \left\| \frac{u}{|x|} \right\|_{L^1} &= \frac{1}{|\mathbb{S}^{n-1}|} \left\| \frac{1}{|x|} \int_{\mathbb{R}^n} \frac{(x-y) \cdot \nabla u(y)}{|x-y|^n} dy \right\|_{L^1} \\ &\leq \frac{1}{|\mathbb{S}^{n-1}|} \sum_{j=1}^n \left\| \frac{1}{|x|} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^n} \nabla u(y) dy \right\|_{L^1} \\ &\leq C \|\nabla u\|_{L^1}. \end{aligned}$$

Moreover, by the same idea, we can give a new proof of a class of Caffarelli-Kohn-Nirenberg inequalities (see [8]) as follows:

(1) Let $n \geq 2$. If $\alpha < 1$, $\beta < \frac{n}{q}$, $0 < \alpha + \beta \leq 1$ and $\frac{1}{q} = 1 + \frac{\alpha+\beta-1}{n}$, then

$$\left\| \frac{u}{|x|^\beta} \right\|_{L^q} \leq C \| |x|^\alpha \nabla u \|_{L^1}. \quad (4.1)$$

(2) If $1 < p < \infty$, $\alpha < \frac{n}{p'}$, $\beta < \frac{n}{q}$, $\alpha + \beta \geq 0$ and $\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta-1}{n}$, $p \leq q < \infty$, then

$$\left\| \frac{u}{|x|^\beta} \right\|_{L^q} \leq C \| |x|^\alpha \nabla u \|_{L^p}. \quad (4.2)$$

Remark 4.1. Take $\alpha = 0$ in the inequality (4.1), we get

$$\left\| \frac{u}{|x|^\beta} \right\|_{L^q} \leq C \|\nabla u\|_{L^1},$$

where $n \geq 2$, $\beta = 1 - n(1 - \frac{1}{q})$ and $1 \leq q < n'$.

Another application is to establish some new weighted div-curl inequalities, and to give a new proof to inequalities of Bourgain and Brezis, Lanzani and Stein, see [4, 5] or [16].

Theorem 4.2. Let $\mathbf{u} \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

(1) Let $\alpha < 1$, $\beta < \frac{3}{q}$, $0 < \alpha + \beta \leq 1$ and $\frac{1}{q} = 1 + \frac{\alpha+\beta-1}{3}$. If $\operatorname{div} \mathbf{u} = 0$, then

$$\left\| \frac{\mathbf{u}}{|x|^\beta} \right\|_{L^q} \leq C \| |x|^\alpha \operatorname{curl} \mathbf{u} \|_{L^1}.$$

(2) If $\operatorname{div} \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^3)$, then

$$\|\mathbf{u}\|_{L^{\frac{3}{2}}} \leq C \left(\sum_{j=1}^3 \|\mathbf{R}_j(\operatorname{div} \mathbf{u})\|_{L^1} + \|\operatorname{curl} \mathbf{u}\|_{L^1} \right). \quad (4.3)$$

(3) Let $1 < p < \infty$, $\alpha < \frac{3}{p'}$, $\beta < \frac{3}{q}$, $\alpha + \beta \geq 0$ and $\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta-1}{3}$. If $p \leq q < \infty$, then

$$\left\| \frac{\mathbf{u}}{|x|^\beta} \right\|_{L^q} \leq C (\| |x|^\alpha \operatorname{div} \mathbf{u} \|_{L^p} + \| |x|^\alpha \operatorname{curl} \mathbf{u} \|_{L^p}).$$

Proof. By Green's representation formula and the vector identity $\operatorname{curl} \operatorname{curl} \mathbf{u} = -\Delta \mathbf{u} + \nabla(\operatorname{div} \mathbf{u})$, we obtain

$$\begin{aligned}
\mathbf{u}(x) &= \Gamma * (-\Delta \mathbf{u})(x) = \int_{\mathbb{R}^3} \Gamma(x-y)(-\Delta \mathbf{u})(y) dy \\
&= \int_{\mathbb{R}^3} \Gamma(x-y)(\operatorname{curl} \operatorname{curl} \mathbf{u} - \nabla(\operatorname{div} \mathbf{u}))(y) dy \\
&= \int_{\mathbb{R}^3} \operatorname{curl}(\Gamma(x-y) \operatorname{curl} \mathbf{u}(y)) - \nabla_y \Gamma(x-y) \times \operatorname{curl} \mathbf{u}(y) - \Gamma(x-y)(\nabla \operatorname{div} \mathbf{u})(y) dy \\
&= \int_{\mathbb{R}^3} -\nabla_y \Gamma(x-y) \times \operatorname{curl} \mathbf{u}(y) + \nabla_y \Gamma(x-y) \operatorname{div} \mathbf{u}(y) dy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} -\frac{x-y}{|x-y|^3} \times \operatorname{curl} \mathbf{u}(y) + \frac{x-y}{|x-y|^3} \operatorname{div} \mathbf{u}(y) dy.
\end{aligned} \tag{4.4}$$

We have used integration by parts twice in the above calculations.

If $p > 1$, by Theorem 1.2 and the following inequality

$$|\mathbf{u}(x)| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} |\operatorname{curl} \mathbf{u}(y)| + \frac{1}{|x-y|^2} |\operatorname{div} \mathbf{u}(y)| dy,$$

we get the third item.

If $p = 1$ and $\operatorname{div} \mathbf{u} = 0$, since $\operatorname{div} \operatorname{curl} \mathbf{u} = 0$, by Theorem 1.5 and the following inequality

$$|\mathbf{u}(x)| \leq \frac{1}{4\pi} \sum_{j=1}^3 \left| \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x-y|^3} \operatorname{curl} \mathbf{u}(y) dy \right|,$$

we get the first item. By Theorem 1.4, Proposition 3.7 and the identity (4.4), we get the second item. \square

Remark 4.3. We replace $\|\operatorname{div} \mathbf{u}\|_{\mathcal{H}^1}$ by $\sum \|R_j(\operatorname{div} \mathbf{u})\|_{L^1}$ in the inequality (4.3). Recently, Xiang and the author [29] have established some related inequalities in smooth bounded domains.

We can generalize Theorem 4.2 to general dimensions.

Theorem 4.4. Let $\mathbf{u} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

(1) Let $n \geq 3$, $\alpha < 1$, $\beta < \frac{n}{q}$, $0 < \alpha + \beta \leq 1$ and $\frac{1}{q} = 1 + \frac{\alpha+\beta-1}{n}$. If $\operatorname{div} \mathbf{u} = 0$, then

$$\left\| \frac{\mathbf{u}}{|x|^\beta} \right\|_{L^q} \leq C \| |x|^\alpha \operatorname{CURL} \mathbf{u} \|_{L^1}.$$

(2) Let $n \geq 3$. If $\operatorname{div} \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^n)$, then

$$\|\mathbf{u}\|_{L^{n'}} \leq C \left(\sum_{j=1}^n \|R_j(\operatorname{div} \mathbf{u})\|_{L^1} + \|\operatorname{CURL} \mathbf{u}\|_{L^1} \right).$$

- (3) Let $n \geq 2$, $1 < p < \infty$, $\alpha < \frac{n}{p'}$, $\beta < \frac{n}{q}$, $\alpha + \beta \geq 0$ and $\frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta - 1}{n}$. If $p \leq q < \infty$, then

$$\left\| \frac{\mathbf{u}}{|x|^\beta} \right\|_{L^q} \leq C(\| |x|^\alpha \operatorname{div} \mathbf{u} \|_{L^p} + \| |x|^\alpha \operatorname{CURL} \mathbf{u} \|_{L^p}).$$

Proof. By Green's representation formula and the identity (2.1), we obtain

$$\begin{aligned} \mathbf{u}(x) &= \int_{\mathbb{R}^n} \Gamma(x-y)(-\Delta \mathbf{u})(y) dy \\ &= \int_{\mathbb{R}^n} \Gamma(x-y)(-\operatorname{div} \operatorname{CURL} \mathbf{u} - \nabla(\operatorname{div} \mathbf{u}))(y) dy \\ &= \int_{\mathbb{R}^n} \nabla_y \Gamma(x-y) \cdot \operatorname{CURL} \mathbf{u}(y) + \nabla_y \Gamma(x-y) \operatorname{div} \mathbf{u}(y) dy \\ &= \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \cdot \operatorname{CURL} \mathbf{u}(y) + \frac{x-y}{|x-y|^n} \operatorname{div} \mathbf{u}(y) dy, \end{aligned} \tag{4.5}$$

where the dot product between a vector \mathbf{v} and a matrix $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)^T$ is defined by $\mathbf{v} \cdot \mathbf{A} = (\mathbf{v} \cdot \mathbf{A}_1, \mathbf{v} \cdot \mathbf{A}_2, \dots, \mathbf{v} \cdot \mathbf{A}_n)$.

If $p > 1$, then by Theorem 1.2, we obtain the third item.

Next we turn to the case $p = 1$ and $\operatorname{div} \mathbf{u} = 0$. For $1 \leq i < j < k \leq n$, set $(f_i, f_j, f_k) = (\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}) \times (u_i, u_j, u_k)$, $f_l = 0$ if $l \neq i, j, k$. Then we have $\operatorname{div} \mathbf{f} = 0$. Applying Theorem 1.5 to \mathbf{f} , then we obtain

$$\left\| \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{x_m - y_m}{|x-y|^n} \left(\frac{\partial u_i}{\partial y_j} - \frac{\partial u_j}{\partial y_i} \right)(y) dy \right\|_{L^q} \leq C \| |x|^\alpha \mathbf{f} \|_{L^1} \leq C \| |x|^\alpha \operatorname{CURL} \mathbf{u} \|_{L^1},$$

for any $1 \leq m \leq n$. Then using the identity (4.5) with $\operatorname{div} \mathbf{u} = 0$, we get the first item. By Theorem 1.4, Proposition 3.7 and the identity (4.5), we get the second item. \square

We can give another proof to the third inequality of Theorem 4.4. We first state the following two facts:

- (1) For any $\mathbf{u} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, it holds that

$$\nabla \mathbf{u} = -\nabla(-\Delta)^{-1} \operatorname{div} \operatorname{CURL} \mathbf{u} - \nabla(-\Delta)^{-1} \nabla(\operatorname{div} \mathbf{u}),$$

- (2) If $1 < p < \infty$ and $-\frac{n}{p} < \alpha < \frac{n}{p'}$, then $|x|^{\alpha p}$ is in the class A_p (see [18, Proposition 1.4.4]).

Using the above two results and the idea of proof given in [15], we can generalize [15, Lemma 2.4] to any dimension $n \geq 2$:

If $1 < p < \infty$ and $-\frac{n}{p} < \alpha < \frac{n}{p'}$, then for any $\mathbf{u} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we have the following weighted inequality for div-curl-grad operators

$$\| |x|^\alpha \nabla \mathbf{u} \|_{L^p} \leq C(\| |x|^\alpha \operatorname{div} \mathbf{u} \|_{L^p} + \| |x|^\alpha \operatorname{CURL} \mathbf{u} \|_{L^p}). \tag{4.6}$$

By Caffarelli-Kohn-Nirenberg inequality (4.2) and the above inequality, we can also obtain the third item of Theorem 4.4. However, this method is not applicable for the case $p = 1$.

Acknowledgements. First, I would like to express my gratitude to my supervisor Prof. Xingbin Pan for guidance and constant encouragement. I also would like to thank Dr. Xingfei Xiang for introducing me some inequalities involving L^1 -norm, Deliang Chen, Yong Zeng and Dr. Huyuan Chen for useful discussions and suggestions. The work was partly supported by the National Natural Science Foundation of China grant no. 11171111 and by Outstanding Doctoral Dissertation Cultivation Plan of Action (PY2015038).

REFERENCES

- [1] C. Amrouche, H. H. Nguyen, *New estimates for the div-curl-grad operators and elliptic problems with L^1 -data in the whole space and in the half-space*, J. Differential Equations. **250** (7), (2011) 3150-3195.
- [2] A. Bensoussan, J. Frehse, *Regularity results for nonlinear elliptic systems and applications*, Applied Mathematical Sciences. **151**, Springer-Verlag, Berlin, 2002.
- [3] J. Bourgain, H. Brezis, *On the equation $\operatorname{div} Y = f$ and application to control of phases*, J. Amer. Math. Soc. **16** (2), (2002) 393-426.
- [4] J. Bourgain, H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris **338**, (2004) 539-543.
- [5] J. Bourgain, H. Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. **9** (2), (2007) 277-315.
- [6] P. Bousquet, P. Mironescu, *An elementary proof of an inequality of Maz'ya involving L^1 vector fields*, Nonlinear elliptic partial differential equations, 59-63, Contemp. Math. **540**, Amer. Math. Soc., Providence, RI, 2011.
- [7] P. Bousquet, P. Mironescu, E. Russ, *A limiting case for the divergence equation*, Math. Z. **274**, (2013) 427-460.
- [8] L. A. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weights*, Compos. Math. **53** (3), (1984) 259-275.
- [9] S. Chanillo, P. L. Yung, *An improved Strichartz estimate for systems with divergence free data*, Comm. Partial Differential Equations. **37** (2), (2012) 225-233.
- [10] F. Demengel, G. Demengel, *Functional spaces for the theory of elliptic partial differential equations*, Universitext, Springer, London, 2012, Translated from the 2007 French original by Reinie Ern .
- [11] P. L. De N poli, I. Drelichman, R. G. Dur n, *On weighted inequalities for fractional integrals of radial functions*, Ill. J. Math. **55** (2), (2011) 575-587.
- [12] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (5), (2012) 521-573.
- [13] R. L. Frank, E. H. Lieb, R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schr dinger operators*, J. Amer. Math. Soc. **21** (4), (2008) 925-950.
- [14] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.
- [15] Q. S. Jiu, Y. Wang, Z. P. Xin, *Global well-posedness of the Cauchy problem of two-dimensional compressible Navier-Stokes equations in weighted spaces*, J. Differential Equations **255** (3), (2013) 351-404.
- [16] L. Lanzani, E. M. Stein, *A note on div curl inequalities*, Math. Res. Lett. **12** (1), (2005) 57-61.
- [17] E. H. Lieb, M. Loss, *Analysis*, Second edition, Graduate Studies in Mathematics. **14**, American Mathematical Society, Providence, RI, 2001.
- [18] S. Z. Lu, Y. Ding, D. Y. Yan, *Singular integrals and related topics*, World Scientific, Singapore, 2007.
- [19] V. Maz'ya, *Bourgain-Brezis type inequality with explicit constants*, In: Interpolation Theory and Applications, 247-252, Contemp. Math. **445**, Amer. Math. Soc., Providence, RI, 2007.
- [20] V. Maz'ya, *Estimates for differential operators of vector analysis involving L^1 -norm*, J. Eur. Math. Soc. **12** (1), (2010) 221-240.
- [21] A. Schikorra, D. Spector, J. Van Schaftingen, *An L^1 -type estimate for Riesz potentials*, accepted for publication in Rev. Mat. Iberoam.
- [22] L. E. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, PhD Thesis, University of Texas at Austin, 2005.
- [23] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. **30**, Princeton University Press, Princeton, N.J., 1970.

- [24] E. M. Stein, G. Weiss, *Fractional integrals on n -dimensional Euclidean space*, J. Math. Mech. **7**, (1958) 503-514.
- [25] E. M. Stein, G. Weiss, *On the theory of harmonic functions of several variables I. The theory of H^p -spaces*, Acta Math. **103**, (1960) 25-62.
- [26] J. Van Schaftingen, *Estimates for L^1 vector fields with a second order condition*, Acad. Roy. Belg. Bull. Cl. Sci. **15** (6), (2004) 103-112.
- [27] J. Van Schaftingen, *Estimates for L^1 vector fields under higher-order differential conditions*, J. Eur. Math. Soc. **10** (4), (2008) 867-882.
- [28] J. Van Schaftingen, *Limiting fractional and Lorentz space estimates of differential forms*, Proc. Amer. Math. Soc. **138** (1), (2010) 235-240.
- [29] X. F. Xiang, Z. B. Zhang, *Hardy-type inequalities for vector fields with vanishing tangential components*, Proc. Amer. Math. Soc. **143** (12), (2015) 5369-5379.

ZHIBING ZHANG: DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, P.R. CHINA;

E-mail address: zhibingzhang29@126.com